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CONCENTRATION FIELD OF PARTICLES EJECTED INTO A NONHOMOGENEOUS ATMOSPHERE BY A MOVING SOURCE

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We obtain and analyze analytical expressions describing the space-time evolution of the density of solid particles ejected into the atmosphere.

Solid or liquid particles ejected as combustion products from aircraft engines are one of the important factors of man's influence on the earth's atmosphere [1]. In solid-fueled engines up to 1/3 of the weight of the exhaust is made up of metal-oxide particles whose typical size is of the orders of a few microns [2, 3]. The purpose of the present paper is to study theoretically the space-time structure of the exhaust trail of particles produced by a moving aircraft.

In the dense layers of the atmosphere the particles are carried along completely by the gaseous components of the exhaust and the distribution of particles is therefore similar to the concentration distributions of the gas components. However it is known that the velocity relaxation length (i.e., the distance the particle falls during the velocity relaxation time) in the important case of large knudsen number is inversely proportional to the density of the medium [4, 5], whereas the transverse size of the exhaust, gas jet is inversely proportional to the square root of the density [6]. Therefore at a certain altitude of flight the size of the cloud of particles begins to exceed the transverse dimensions of the region occupied by the gas components of the exhaust. In the first approximation, for altitudes greater than this critical height one can assume that the distribution of particles in the exhaust trail is formed as a result of ejection of particles into a nonmoving atmosphere from a moving point source, which corresponds to the near-nozzle flow region determining the initial velocity of the particles.

Then the equation of motion for the distribution function f of particles of a given size is:

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{x}}+\frac{\partial}{\partial \mathbf{v}}((\mathbf{g}+\mathbf{a})f)=0. \tag{1}$$

The drag force on a particle in the medium is assumed to be proportional to its velocity:

$$\mathbf{a}=-\gamma\cdot\mathbf{v}, \tag{2}$$

where the dependence of the reciprocal of the relaxation time on height is approximated as an exponential:

$$\gamma=\gamma(x_3)=\gamma(0)\exp(-x_3H^{-1}), \tag{3}$$

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and $\gamma(0)$ depends on the size of the particle.

The main advantage of the approximation (2) is that it allows one to obtain an analytical solution of the problem. The actual drag force acting on a particle moving in the atmosphere at supersonic speeds (usually) in the free-molecular flow regime differs from (2). However the main qualitative features of the results obtained using (2) will obviously continued to hold for a more exact form of a drag force on the particle.

Solving (1) with the initial condition

$$f(0, \mathbf{x}, \mathbf{v}) = \delta(\mathbf{x} - \mathbf{x}^0) F(\mathbf{v}) \tag{4}$$

and integrating the result with respect to velocity, we obtain the spatial density distribution of particles from an instantaneous point source:

$$\rho_0(t, \mathbf{x} \mid 0, \mathbf{x}^0) = \frac{\gamma}{\Gamma} \frac{1 + \Gamma \int_0^t B(t')\,dt'}{A^2(t)\left[tB(t) - \gamma \int_0^t t'B(t')\,dt'\right]} \times \\ \times F\left(\frac{x_1 - x_1^0}{A(t)}, \frac{x_2 - x_2^0}{A(t)}, H(\Gamma - m)\right), \tag{5}$$

where

$$A(t) = \int_0^t dt' \left[1 + \Gamma \int_0^{t'} B(t'')\,dt''\right]^{-1}; \tag{6}$$

$$\Gamma = \gamma(x_3^0); \tag{7}$$

$$B(t) = \exp(mt + 0{,}5gt^2H^{-1}), \tag{8}$$

and m is determined from the condition

$$\gamma\left(1 + \Gamma \int_0^t B(t')\,dt'\right) = \Gamma B(t). \tag{9}$$

The density of particles created by a source of mass flow rate $Q(t)$ moving along the trajectory $\mathbf{x}_s(t)$ has the form

$$\rho(t, \mathbf{x}) = \int_{-\infty}^t dt' Q(t') \rho_0(t, \mathbf{x} \mid t', \mathbf{x}_s(t')). \tag{10}$$

For further calculations we make the simplifying assumption that all particles are ejected into the atmosphere with a single value of the component of the velocity V_a along the trajectory:

$$F(\mathbf{v}) = F_\perp(v_1, v_3\cos\alpha - v_2\sin\alpha)\,\delta(v_2\cos\alpha + v_3\sin\alpha - \Delta V), \tag{11}$$

where $F_\perp$ is the velocity distribution function in the plane perpendicular to the trajectory, and

$$\Delta V = V_s - V_a. \tag{12}$$

As an example, we will assume the function F has the form

$$F_\perp(a_1, a_2) = (2\pi v_\perp^2)^{-1} \exp[-0{,}5(a_1^2 + a_2^2) v_\perp^{-2}]. \tag{13}$$

Substituting (5) and (11) into (10) and using the fact that V_s and α can depend on time, we obtain

$$\rho(t, \mathbf{x}) = Q(\tau)\left(\frac{\partial P}{\partial \tau}\right)^{-1} F_{\perp}\left\{\frac{x_1 - x_{1s}^0}{A(t-\tau)}, \quad \frac{H[\Gamma(\tau) - m] - \Delta V \sin\alpha}{\cos\alpha(\tau)}\right\} \times$$

$$\times \frac{\gamma}{\Gamma(\tau) A^2(t-\tau)} \frac{1 + \Gamma(\tau)\int\limits_0^{t-\tau} B(t')\,dt'}{(t-\tau)B(t-\tau) - \gamma\int\limits_0^{t-\tau} t'B(t')\,dt'}, \quad (14)$$

where

$$P(\tau) = A^{-1}(t-\tau)\left[x_2 - x_{2s}^0 - \int\limits_0^{\tau} V_s(t')\cos\alpha(t')\,dt'\right]\cos\alpha(\tau) + \quad (15)$$

$$+ H[\Gamma(\tau) - m]\sin\alpha(\tau) - \Delta V(\tau);$$

$$\Gamma(\tau) = \Gamma_0 \exp\left[-\int\limits_0^{\tau} V_s(t')\sin\alpha(t')\,dt'\right]; \quad (16)$$

$$\Gamma_0 = \gamma(x_{3s}^0); \quad (17)$$

$\mathbf{x}_s^0$ is the position of the aircraft at t = 0 and the quantities τ and m are found from the equations

$$\gamma\Gamma^{-1}(\tau) = B(t-\tau) - \gamma\int\limits_0^{t-\tau} B(t')\,dt', \quad (18)$$

$$P(\tau) = 0. \quad (19)$$

We consider first the case when the height is sufficiently small, such that the particle velocity relaxation length is much smaller than the scale height H of the atmosphere. After a sufficiently long time the structure of the exhaust trail at a given point will be determined by particles ejected into the atmosphere in the neighborhood of that point within an altitude range much smaller than H. Usually the trajectory parameters and the mass flow rate vary only weakly in this altitude range. Therefore V_s, α, and Q can be considered as constants and equal to their values at the point $\mathbf{x}_s^0$, which is chosen to be a much closet than H to the region of interest. Then for times

$$\Gamma_0^{-1} \ll t \ll H\Gamma_0 g^{-1} \quad (20)$$

it follows from (14)-(19) that

$$\rho(t, \mathbf{x}) = Q\Gamma_0^2 V_s^{-1} F_{\perp}\{(x_1 - x_{1s}^0)\Gamma_0, \quad r\Gamma_0 + g(t - t')\cos\alpha\}, \quad (21)$$

where

$$r = (x_3 - x_{3s}^0)\cos\alpha - (x_2 - x_{2s}^0)\sin\alpha, \quad (22)$$

$$t' = V_s^{-1}[(x_2 - x_{2s}^0)\cos\alpha + (x_3 - x_{3s}^0)\sin\alpha]. \quad (23)$$

It was assumed in (21) that the velocity due to gravity and the attained value of the transverse coordinate r are both relatively small:

$$g\Gamma_0^{-1} \ll V_s, \quad r \ll H. \quad (24)$$

In addition, small terms $\sim H^{-1}$ are omitted in (21).

It is evident from (21) that at times larger than Γ_0^{-1} the concentration of particles is characterized by a steady transverse distribution whose center of gravity falls uniformly. The transverse dimensions of the exhaust can be estimated using (13):

$$R \sim v_{\perp} \Gamma_0^{-1}. \tag{25}$$

The evolution of the density distribution is easily followed with the help of (5)-(9), using the fact that in this case we have the inequality

$$(\gamma - \Gamma)\Gamma^{-1} \ll 1. \tag{26}$$

Neglecting for simplicity the force of gravity, which is valid for times

$$t < t^* \sim \min(v_3^* g^{-1}, \quad \sqrt{Hg^{-1}}), \tag{27}$$

we find

$$\rho_0(t, \mathbf{x} \mid 0, \mathbf{x}^0) = \left[\frac{\Gamma}{1 - \exp(-\Gamma t)}\right]^3 F\left\{\frac{(\mathbf{x} - \mathbf{x}^0)\Gamma}{1 - \exp(-\Gamma t)}\right\}, \tag{28}$$

and hence for times $t \ll \Gamma^{-1}$ the density evolves as a result of inertial dispersion of the particles and a steady-state distribution is established after a time $\sim \Gamma^{-1}$.

In practice the most critical condition for the applicability of (21) is that the transverse dimensions of the exhaust trail be sufficiently small

$$R \ll H. \tag{29}$$

Using the data of [4] for Al_2O_3 particles of radius 5 μm, we obtain that for $v_{\perp} \sim 1$ km/sec the condition (29) is satisfied up to altitudes of about 75 km. We have $R \sim 1$ km at an altitude of 75 km.

It is difficult to use (14) analytically for intermediate altitudes where $R \sim H$. However it is possible to obtain a number of features of the space-time distribution of particles using the moments of the function (14). We consider the case of horizontal flight ($\alpha = 0$) with a constant velocity equal to the velocity of the exhaust ($\Delta V = 0$) and a constant mass flow rate. The distribution of particles in the plane perpendicular to the flight trajectory can be characterized in terms of the moments

$$M_n = \int \rho(t, x_1, x_{2s}^0, x_3)(x_3 - x_{3s}^0)^n \, dx_1 dx_3 = \\ = \frac{Q}{V_s} \frac{H}{\sqrt{2\pi v_{\perp}^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{H^2(\Gamma_0 - m)^2}{2v_{\perp}^2}\right]\left[-H \ln \frac{\gamma(m)}{\Gamma_0}\right]^n dm, \tag{30}$$

where $\gamma(m)$ is determined by (9) and we have used (13) for $F_{\perp}$. Evaluating (30) approximately at times

$$\Gamma_0^{-1} \ll t \ll t^*, \tag{31}$$

we obtain with the help of the assumption (29):

$$\bar{x}_3 - x_{3s}^0 = \frac{M_1}{M_0} \simeq \frac{R^2}{2H} + C(t)\frac{R^3}{H^2}\Gamma_0 t, \tag{32}$$

$$\overline{(x_3 - \bar{x}_3)^2} = \frac{M_2}{M_0} - \frac{M_1^2}{M_0^2} = R^2\left[1 + \frac{7}{16}\frac{R^2}{H^2} + 2C(t)\frac{R^3}{H^3}(\Gamma_0 t)^2\right], \tag{33}$$

$$C(t) = (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{H^2}{2R^2} - \frac{H^2}{R^2\Gamma_0 t}\right]. \tag{34}$$

We see from (32) that even at the initial time the center of gravity of the trail is displaced upward with respect to the trajectory by the quantity $R^2/2H$, which is due to an asymmetry in the dispersion of particles in the upward and downward directions. The origin of the time-dependent terms in (32) and (33) can be understood by analyzing the motion of an individual particle with initial velocity v_3^0 in an atmosphere with an exponential density distribution (and assuming $g = 0$):

$$x_3(t) = x_3^0 - H \ln \frac{\exp \beta t}{1 + \frac{\Gamma}{\beta}(\exp \beta t - 1)},$$

where

$$\beta = \Gamma - v_3^0 H^{-1}. \qquad (36)$$

We see from (35) that when $\beta < 0$, i.e., when the initial upward velocity of the particle is sufficiently large, the particle, after 'leaving' the atmosphere, then moves with a constant residual velocity $v_0^3 - H\Gamma$. For (32) and (33) this means that particles with sufficiently high initial vertical velocities which are permitted by the distribution function (13) escape upward, and hence the center of gravity of the density distribution (32) shifts with time and its vertical size (33) increases. It is obvious that this effect will take place as long as we can neglect the force of gravity.

From (33), and also from the fact that the inertial stage of expansion ($t \leqslant \Gamma_0^{-1}$) is unsteady, we conclude that when

$$v_\perp \Gamma_0^{-1} \sim H \qquad (37)$$

the exhaust trail becomes significantly unsteady. For Al_2O_3 particles of radius 5 µm and $v_\perp \sim 1$ km/sec, the trail can become unsteady at altitudes greater than 90 km.

We transform to the limiting case of very large altitudes where air resistance can be neglected. Analysis of (5) with the conditions

$$\gamma \simeq 0,\ \Gamma \simeq 0,\ \gamma/\Gamma = \exp((x_3^0 - x_3)/H) \qquad (38)$$

gives

$$\rho_0(t,\ \mathbf{x}|0,\ \mathbf{x}^0) = t^{-3} F\{(\mathbf{x} + 0{,}5\mathbf{g}t^2 - \mathbf{x}^0)\,t^{-1}\}. \qquad (39)$$

We obtain the density of particles behind a continuous source for the case of linear, constant-acceleration motion of the aircraft (α = const):

$$V_s = V + at, \qquad (40)$$

where V and a are the velocity and acceleration at the point x_s^0. Using (38) and (40), we obtain from (14)

$$\rho(t,\ \mathbf{x}) = \frac{Q}{(t-\tau)^2 [V_a + (g \sin\alpha + a)(t-\tau)]} \times \times F_\perp \left\{ \frac{x_1 - x_{1s}^0}{t-\tau},\ \frac{r + 0{,}5g(t-\tau)^2 \cos\alpha}{t-\tau} \right\}, \qquad (41)$$

where

$$t - \tau = [-V_a + \sqrt{V_a^2 + 2L(a + g\sin\alpha)}]\,(a + g\sin\alpha)^{-1}; \qquad (42)$$

$$L = Vt + 0{,}5at^2 - (x_2 - x_{2s}^0)\cos\alpha - (x_3 - x_{3s}^0)\sin\alpha. \qquad (43)$$

Using (41) and (13), we estimate the thickness of the exhaust trail in the 1 direction. For the point (x_{2S}^0, x_{3S}^2) we have

$$R \sim v_\perp [-V_a + \sqrt{V_a^2 + 2(g\sin\alpha + a)(Vt + 0{,}5at^2)}]\,(g\sin\alpha + a)^{-1}. \qquad (44)$$

For small times

$$R \sim v_\perp (Vt + 0{,}5at^2)\,V_a^{-1}. \qquad (45)$$

This equation shows that the particles are contained within a cone of angle $\tan^{-1}(v_\perp/v_a)$ moving with a constant acceleration.

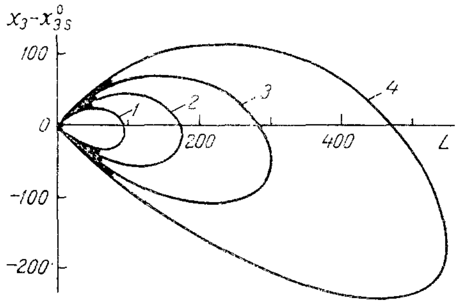

Fig. 1. Lines of constant particle density in the plane $x_1 = x^0_{1S}$: 1) $\rho/\bar{\rho} = 10^{-3}$; 2) $3\cdot10^{-4}$; 3) 10^{-4}; 4) $3\cdot10^{-5}$, where $\rho = Q[2\pi v_\perp^2 V_a]^{-1} V_a^2 \bar{L}^{-2}$; $\bar{L}$ is the distance from the aircraft numerically equal to V_a (x_3-x_{3S} and L are in km).

As an example of the spatial distribution of particles at large altitudes we show in Fig. 1 the lines of constant density calculated from (41) and (13) for the case of horizontal flight with a constant velocity.

NOMENCLATURE

f, velocity distribution function of particles (depends on position and time); t, time; x, position vector referenced to an earth-fixed coordinate system; x_1 and x_2 are in the horizontal plane with $x_\perp$ perpendicular to the trajectory plane; x_3 is the vertical coordinate; x_s, position vector of the aircraft (source of particles); v, particle velocity; a, particle acceleration due to braking in the atmosphere; g, acceleration due to gravity; γ, reciprocal of the velocity relaxation time; H, scale height of the atmosphere; $F(v)$, initial velocity distribution function; B(t), defined by (8); Q(t), mass flow rate of particles; $\rho(t, x)$ density of particles created by a continuous source; $\rho_0(t, x|0, x^0)$ density of particles created by an instantaneous point source at the point x^0; α, angle of the trajectory to the horizontal; V_S, velocity of the aircraft; V_a velocity of the exhaust gases at the nozzle exit plane; $F_\perp$, initial distribution function for the transverse components of the velocity; R, transverse dimension of the exhaust trail; t^*, time up to which one can neglect the effect of gravity; v_3^*, characteristic value of the vertical velocity; $v_\perp$, characteristic value of the initial transverse velocity; Γ, determined by (17); L. determined by (43).

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